Mittag-Leffler state estimator design and synchronization analysis for fractional order BAM neural networks with time delays

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Abstract

This paper deals with the extended design of Mittag-Leffler state estimator and adaptive synchronization for fractional order BAM neural networks (FBNNs) with time delays. By the aid of Lyapunov direct approach and Razumikhin-type method a suitable fractional order Lyapunov functional is constructed and a new set of novel sufficient condition are derived to estimate the neuron states via available output measurements such that the ensuring estimator error system is globally Mittag-Leffler stable. Then, the adaptive feedback control rule is designed, under which the considered FBNNs can achieve Mittag-Leffler adaptive synchronization by means of some fractional order inequality techniques. Moreover, the adaptive feedback control may be utilized even when there is no ideal information from the system parameters. Finally, two numerical simulations are given to reveal the effectiveness of the theoretical consequences.

Keywords. Mittag-Leffler synchronization; BAM neural networks; Fractional order; Time-delays; Adaptive feedback control.

1 Introduction

Fractional order differential equation is a natural expansion of traditional integer order differential equations, dating from around three hundred years prior. Many of the researchers threw themselves into fractional order differential equations for plenty of years. However, due to lack of its application history and its complexity in numerous areas, and also for a long time fractional order differential equations were extensively studied within the field of mathematics. Until recently, the facts proved that the principle of fractional order calculus are an excellent instrument in an epidemic model [1], financial system[7], heat conduction[11], circuit systems [12], market dynamics[17], biological model[23], dielectric polarization [27] and so forth. Compared to traditional integer order dynamical modeling, fractional order dynamical modeling is more advanced, for this reason, it has infinite memory property and more degree of freedom, for more details, one can refer to the monograph of [16, 28]. Since stability issues are more effective to measure any dynamical behaviors. Up to now, there are numerous kinds of stability problems of fractional order dynamical system that are available in the existing literature.

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For example, in Ref [29], V.N. Phat et al. discussed the finite time stability of nonlinear fractional order system by means of Gronwall inequality approach. In [20], Liu et al introduced the asymptotic stability of the fractional order nonlinear system based on the Riemann Liouville operator. In [50], Yang et al. investigated the Mittag-Leffler stability of fractional order nonlinear system by using the Lyapunov method, S-procedure and impulsive differential equations.

A form of bidirectional associative memory (BAM) neural networks was firstly proposed by Kosko, which has a special structure of connection weights. As is known to all, BAM neural networks are extended from one-layer auto-associative Hebbian-correlate to a two-layer pattern-matched hetero associative circuit [22]. The major advantage of this extension is, it recalls and stores pattern pairs regarded as the bidirectionally stable states. In contrast to other types of neural networks, BAM neural networks is a composition of neurons which are organized in two layers, such as P-layer and Q-layer. The neurons in one layer are completely interconnected to the neurons only in the different layer, at the same time there is no interconnection among neurons in the similar layer. Nonlinear dynamical systems are used in many practical situations, where time delays are inevitable because of the finite speed of signal transmission and the network parameter fluctuations of the hardware implementation. Therefore, the study of nonlinear delayed dynamical behaviors are meaningful. In the past few years, the study of the dynamical behavior of delayed BAM neural networks grown to be a hot research topic and it is efficaciously performed in different fields such as combinational optimization, associative memory, parallel computing, automatic control and so on. In the meantime, the dynamic behavior of BAM neural networks creates great interests among the researchers and many of the excellent results have been reported, see Ref [31, 32, 33]. Substantially, in recent years a developing issue on dynamical behavior of fractional order BAM neural networks such as stability[37, 42], stabilization [47], bifurcations[14] and synchronization [3, 38]. In application perspective, it is difficult to completely acquire the state information of all neurons due to their entangled structure. Mainly the state components of the neural network model are generally unknown or no longer to be had for direct measurement. That is, the neuron states are not regularly completely available within the network outputs. So the issues in state estimation of neural networks will become an essential topic in neural networks in practice. As of late, a few researchers have determined numerous outcomes on the outline of state estimator for a different sort of integer order neural network systems [18, 24, 35, 43] and the state estimation of BAM neural networks has also been reported in [2, 36, 39]. For instance, the issues of state estimation for a class of discrete-time BAM neural networks with delay was analyzed via LyapunovKrasovskii functional together with LMI techniques in [2]. Recently, Ratnavelu et al. [36] has investigated the problem of state estimation for integer order fuzzy cellular BAM neural networks with leakage delay and unbounded distributed delays by means of Lyapunov-Krasovskii functional and LMI techniques. Sakthivel [39] et al. also studied the state estimator design for integer order BAM neural networks with constant leakage delays and time-varying distributed delays via LyapunovKrasovskii functional together with free-weighting matrix technique. Contrast with the state estimation of integer-order BAM neural networks, the state estimation of FBNNs will deliver more accurate neurons state estimation to advantage the application of neural networks. Till now, only very little attention has been paid to fractional order state estimation of neural networks with (or without) time delays [4, 44]. But there is no attention has been paid to state estimation of fractional order BAM neural networks.

The drive-response concept of complete synchronization for a chaotic system has grown to be an active research area, was first proposed by Pecora and Carroll in [30]. The author of [30], proposed the response system which affects the behavior of the drive system but the drive system doesn't depend on the response system. i.e generated signal in driver sent over a channel to the responder, which uses this signal synchronizes itself with the driver. Since they have been fruitfully applied to the area of secure communication, fault diagnosis, biological systems, especially real world neural network models. Up to now, many authors investigate the various sorts of synchronization, such as exponential synchronization, Mittag-Leffler synchronization, Mittag-Leffler projective synchronization, asymptotic synchronization, finite time synchronization, projective synchronization, lag synchronization, quasi-uniform synchronization and $O(t^{-\alpha})-$ synchronization, see Ref [6, 8, 21, 19, 25, 26, 40, 46, 48, 49, 51, 52, 55]. Lots of effective control strategies have been adopted to synchronize in fractional order neural networks, including linear feedback control, adaptive feedback control, impulsive control, sliding mode control, non-fragile control and so on. In the existing literature on synchronization of fractional order BAM neural networks, most of them targeted on the linear feedback controller/delayed feedback controller/impulsive controller. For example, The authors of [9] studied the global Mittag-Leffler synchronization of fractional order delayed BAM neural networks by impulsive control and state feedback control. In [48], the authors discussed the finite time synchronization analysis of fractional order memristor based BAM neural networks with time delays via simple linear feedback controller and the global Mittag-Leffler synchronization of BAM neural networks under the delayed feedback control strategy was investigated in [51]. Under the impulsive controller, the authors [54], addressed the exponential synchronization of fractional order BAM neural networks with different impulsive effects by using Mittag-Leffler functions and average impulsive interval definitions. Since the control gains of the linear feedback controller are very high, which is a kind of dissipating in practice. However, the adaptive controller can avoid the high feedback control gains due to the fact that, it may self-adjust the coupling strengths. Hence, adaptive feedback control is more effective, in comparison to the linear feedback controller. Therefore, the study of an adaptive feedback control strategy for synchronization of fractional order delayed BAM neural networks (FBNNs) is in great demand. Nonetheless, till now, there are very few or even no published works at the issues of Mittag-Leffler state estimator design and adaptive synchronization for FBNNs with time delays. Therefore, it is of great importance to fill this gap. In addition to, it is important to pointed out that our result is true for exponential state estimator and an adaptive exponential synchronization for integer order BAM neural networks with time delays and this works has not been considered yet.

Prompted via the above discussion, in this letter we study the Mittag-Leffler state estimator design and synchronization analysis for FBNNs with time delay, the problem remains open and is not report in existing works literature. Consequently, we can try to remedy this hard and essential problem. In [6, 9, 15, 45, 51], the authors provided the definition of Mittag-Leffler stability and Mittag-Leffler synchronization of fractional-order neural networks. In [51], the authors presented the Mittag-Leffler synchronization of BAM neural networks. Motivated by these definitions, we introduce the definitions of Mittag-Leffler state estimator and adaptive Mittag-Leffler synchronization of FBNNs. The main challenge and contributions of this letter are embodied in the following aspects: (1). In the light of the bilayer structure of FBNNs, global Mittag-Leffler state estimator and global Mittag-Leffler synchronization are first time presented. (2). By utilizing a proper Lyapunov functional, some inequalities and Razumikhin condition, the novel algebraic sufficient conditions are obtained to ensure the estimator error system is globally Mittag-Leffler state estimator in the form of linear matrix inequality. (3). The novel adaptive feedback controller is designed for the FBNNs and the proposed controller is different from [34, 38], while a new type of fraction order inequality is proposed, which helps to achieve the global Mittal-Leffler synchronization goal. (4). The proposed works are new that fill a few gaps in the prevailing works and our outcomes generalize and enhance the ones in present literature.

The rest of this paper is prepared as follows: In Section 2, some necessary fractional order definitions are listed. Further, a few assumptions and Mittag-Leffler state estimator definitions together with a few beneficial lemmas needed in this paper are provided. The main theoretical consequences are derived in Section 3. In Section 4, two numerical examples and their simulations are given to illustrate the effectiveness of the acquired results.

Nomenclature. In this letter, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^m$ denotes that the set of all natural numbers, real numbers and m-D Euclidean space, respectively, and $\mathbb{R}^{m\times m}$ represent the set of all $m \times m$ real matrices. Let $q = (q_1, ..., q_m)^T \in \mathbb{R}^m$ represents a column vector, where superscript T stands for the transpose operator. The two norm of vector q is defined by $\|q\|_2 = \sqrt{q_1^2 + ... + q_m^2}$. $\Upsilon > 0(\Upsilon < 0)$ suggest that Υ is positive definite (negative definite), while $\tilde{\lambda}_{\max}(\Upsilon)$ and $\tilde{\lambda}_{\min}(\Upsilon)$ delegate the maximum and minimum values eigenvalue of Υ. For $\eta > 0$, $\mathcal{C}([-\eta, 0], \mathbb{R}^m)$ represents the family of continuous function from $[-\eta, 0]$ to $\mathbb{R}^m$. The symbol $*$ involves the convolution operator.

2 Preliminaries

In this part, we will display the basic definition's, system description, lemmas of fractional order derivative and assumptions.

2.1 *Fractional order derivative concept and tools*

Definition 2.1 *[28]. The fractional order integral of an integrable function $p(t)$ is defined as*

$$I_{t_0}^{\lambda}\, p(t) = \frac{1}{\Gamma(\lambda)} \int_{t_0}^{t} (t-\omega)^{\lambda-1} p(\omega)\, \mathrm{d}\omega,\ t \geq 0,$$

where $\lambda \in \mathbb{R}^+$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 *[16, 28]. The Caputo type of fractional-order derivative of order $\lambda \in (m-1, m)$, $m \in \mathbb{Z}^+$ for a function $p(t)$ is defined by*

$$D_{t_0,t}^{\lambda}\, p(t) = \frac{1}{\Gamma(m-\lambda)} \int_{t_0}^{t} \frac{p^{(m)}(\omega)}{(t-\omega)^{\lambda-m+1}}\, \mathrm{d}\omega,\ t \geq t_0.$$

In addition, the following properties about Caputo fractional-order derivative are given.

Property 1.[16] $D^{\lambda}\zeta = 0$, where ζ is any constant.

Property 2.[16] For any constant α and β, the linearity of Caputo fractional order derivative gives

$$D^{\lambda}\Big(\alpha p(t) + \beta q(t)\Big) = \alpha D^{\lambda} p(t) + \beta D^{\lambda} q(t).$$

Mittag-Leffler function is generalization of exponential function, which is usually used to describe the solutions of fractional order dynamical behaviors.

Definition 2.3 *[16, 28] A two-parameter Mittag-Leffler function is listed as follows:*

$$\mathcal{E}_{\lambda,\chi}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\lambda j + \chi)},$$

where $\lambda > 0$, $\chi > 0$ and $z \in \mathbb{C}$. For $\chi = 1$, its one parameter Mittag-Leffler function it is shown as

$$\mathcal{E}_{\lambda,1}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\lambda j + 1)}.$$

In particular, one obtains $\mathcal{E}_{1,1}(z) = \exp(z)$ for $\lambda = \chi = 1$. Additionally, The Laplace transform of two parameter Mittag-Leffler function is

$$\mathcal{L}\{t^{\chi-1}\mathcal{E}_{\lambda,\chi}(-\Phi t^{\alpha})\} = \frac{s^{\lambda-\chi}}{s^{\lambda}+\Phi},\ (\mathfrak{Re}(s) \geq |\Phi|^{\frac{1}{\lambda}}),$$

where t and s are, respectively, the variables in the domain and Laplace domain, while $\mathcal{L}\{\cdot\}$ stands for the Laplace transform.

Remark 2.4 *Let $0 < \lambda \leq 1$ and $t \in \mathbb{R}$, we have $\mathcal{E}_{\lambda,\lambda}(t) > 0$, $\mathcal{E}_{\lambda,1} > 0$ and $\frac{d}{dt}\mathcal{E}_{\lambda,\lambda}(t) > 0$.*

Remark 2.5 *The initial values of Caputo derivative can be expressed the integer order terms, which looks like same as initial values of integer order differential equations. Moreover, these operator satisfies the linearity property. Hence the Caputo derivative operator has more effective tool compared to Riemann Liouville operator and it has more applicable to real world-problems, for more details see [10, 28]. Throughout in this letter, we deal with Caputo fractional order derivative operator.*

In order to obtain the Mittag-Leffler state estimator and Mittag-Leffler synchronization results, we provide a following Lemma's as follows:

Lemma 2.6 *[45] For $0 < \lambda < 1$, the continuous function $x(t)$ is defined on the positive interval $[0, +\infty)$, then there exist a positive scalar $\Phi_1 > 0$ and $\Phi_2 \geq 0$ such that*

$$D_{0,t}^{\lambda}\, x(t) \leq -\Phi_1 x(t) + \Phi_2,\ t \geq 0,$$

then

$$x(t) \leq x(0)\mathcal{E}_{\lambda}\big(-\Phi_1 t^{\lambda}\big) + \Phi_2 t^{\lambda}\mathcal{E}_{\lambda,\lambda+1}\big(-\Phi_1 t^{\lambda}\big),\ t \geq 0.$$

The following Lemma 2.7 is an extension of the previous Lemma 2.6.

Lemma 2.7 *For $0 < \lambda < 1$, the two continuous functions $x(t)$ and $y(t)$ are defined on the positive interval $[0, +\infty)$, $0 < l \leq y(t) \leq L$ and satisfy*

$$D_{0,t}^{\lambda}\,\big[x(t) + y(t)\big] \leq -\Phi_1 x(t) + \Phi_2,\ t \geq 0, \tag{1}$$

then there exist $\mathcal{T} > 0$ such that

$$x(t) \leq \big[x(0) + y(0)\big]\mathcal{E}_{\lambda}\big(-\Phi_1 t^{\lambda}\big) + \Phi_2 t^{\lambda}\mathcal{E}_{\lambda,\lambda+1}\big(-\Phi_1 t^{\lambda}\big),\ t \geq \mathcal{T},$$

where $\Phi_1 > 0,\ \Phi_2 \geq 0,\ l$ and L are scalars.

Proof. By (1), it follows there exist a non negative function $J(t)$ satisfies

$$D_{0,t}^{\lambda}\,\big[x(t) + y(t)\big] + J(t) = -\Phi_1 x(t) + \Phi_2,\ t \geq 0, \tag{2}$$

Making Laplace transform on (2), one has obtain

$$s^{\lambda}\big[x(s) + y(s)\big] - s^{\lambda-1}\big[x(0) + y(0)\big] + J(s) = -\Phi_1 x(s) + \frac{\Phi_2}{s},\ t \geq 0,$$

where $x(s) = \mathcal{L}\big(x(t)\big),\ y(s) = \mathcal{L}\big(y(t)\big)$ and $J(s) = \mathcal{L}\big(J(t)\big)$. It follows that

$$x(s) = \frac{s^{\lambda-1}\big[x(0) + y(0)\big] - J(s) - s^{\lambda}y(s) + \frac{\Phi_2}{s}}{s^{\lambda} + \Phi_1}. \tag{3}$$

Next by making a inverse Laplace transform of (3), then the unique solution of (2) is the following form:

$$\begin{aligned} x(t) \quad = \quad & \big[x(0) + y(0)\big]\mathcal{E}_{\lambda}\big(-\Phi_1 t^{\lambda}\big) - J(t) * t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big) \\ & -y(t) * \Big[1 - \Phi_1 t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big)\Big] + \Phi_2 t^{\lambda}\mathcal{E}_{\lambda,\lambda+1}\big(-\Phi_1 t^{\lambda}\big),\ t \geq 0, \end{aligned}$$

where $*$ is the convolution operator. Next, we will to prove that, there exist a $\mathcal{T} > 0$ such that $y(t) * \Big[1 - \Phi_1 t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big)\Big] \geq 0$. Then, assume that $0 < t_1 < t_2$ and by taking $t_2 \leq \mathcal{T}$, such that

$$\frac{\Phi_1}{\Gamma(\lambda)t_1^{\lambda-1}} = 1 \text{ and } \left[t_2 - \frac{\Phi_1 t_2^{\lambda}}{\Gamma(\lambda+1)}\right] l \geq \big(L - l\big)t_1\Big[\frac{1}{\lambda} - 1\Big]. \tag{4}$$

From Remark 2.4, it follows that $\Phi_1 t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big) \leq \frac{\Phi_1}{\Gamma(\lambda)t_1^{\lambda-1}},\ t \geq 0$ and

$$\begin{aligned}
y(t) * \Big[1-\Phi_1 t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big)\Big] &= \int_0^t y(t-\omega)\Big[1-\Phi_1\omega^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1\omega^{\lambda}\big)\Big]\,\mathrm{d}\,\omega \\
&\geq \int_0^t y(t-\omega)\left[1-\frac{\Phi_1}{\Gamma(\lambda)}\omega^{\lambda-1}\right]\mathrm{d}\,\omega \\
&\geq L\int_0^{t_1}\left[1-\frac{\Phi_1}{\Gamma(\lambda)}\omega^{\lambda-1}\right]\mathrm{d}\,\omega + l\int_{t_1}^{t_2}\left[1-\frac{\Phi_1}{\Gamma(\lambda)}\omega^{\lambda-1}\right]\mathrm{d}\,\omega \\
&= L\left[t_1-\frac{\Phi_1}{\lambda\Gamma(\lambda)}t_1^{\lambda}\right]+l\left[t_2-\frac{\Phi_1}{\lambda\Gamma(\lambda)}t_2^{\lambda}-t_1+\frac{\Phi_1}{\lambda\Gamma(\lambda)}t_1^{\lambda}\right] \\
&= l\left[t_2-\frac{\Phi_1}{\Gamma(\lambda+1)}t_2^{\lambda}\right]+\Big[L-l\Big]t_1\left[1-\frac{\Phi_1}{\lambda\Gamma(\lambda)t_1}t_1^{\lambda}\right]
\end{aligned}$$

Then by using Eq.(4), we obtain

$$\begin{aligned}
y(t) * \Big[1-\Phi_1 t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big)\Big] &= l\left[t_2-\frac{\Phi_1}{\Gamma(\lambda+1)}t_2^{\lambda}\right]-\big[L-l\Big]t_1\left[\frac{1}{\lambda}-1\right] \\
&\geq 0
\end{aligned}$$

for any $t \geq t_2$. Otherwise, $J(t),\ t^{\lambda-1}$ and $\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big) \geq 0$ are non negative functions, which is defined on $[0,+\infty)$, thus $J(t) * t^{\lambda-1}\mathcal{E}_{\lambda,\lambda}\big(-\Phi_1 t^{\lambda}\big) \geq 0,\ t \geq 0$. Therefore

$$x(t) \leq \big[x(0)+y(0)\big]\mathcal{E}_{\lambda}\big(-\Phi_1 t^{\lambda}\big)+\Phi_2 t^{\lambda}\mathcal{E}_{\lambda,\lambda+1}\big(-\Phi_1 t^{\lambda}\big),\ t \geq \mathcal{T}.$$

Hence, this proof of Lemma has been completed.

2.2 *Model description*

We consider a class of fractional order BAM neural networks with delay as follows:

$$\begin{cases} D^{\lambda}p_j(t) = -a_j p_j(t)+\sum_{k=1}^{m} b_{kj}g_k(q_k(t))+\sum_{k=1}^{m} c_{kj}g_k(q_k(t-\eta))+G_j, \\ D^{\lambda}q_k(t) = -u_k q_k(t)+\sum_{j=1}^{n} v_{jk}h_j(p_j(t))+\sum_{j=1}^{n} w_{jk}h_j(p_j(t-\eta))+H_k, \end{cases} \tag{5}$$

$j = 1,2,...,n,\ k = 1,2,...,m,$ where $\lambda \in (0,1)$, $D^{\lambda}p(\cdot)$ is the Caputo fractional order derivative of $p(\cdot)$ from 0 to t, namely $D^{\lambda}_{0,t}$; There are twin layer such as P-Layer and Q-Layer in the fractional order neural network systems such as $P = \{p_1,...,p_n\}$ and $Q = \{q_1,...,q_m\}$; $p_j(t)$ and $q_k(t)$ are the state vectors of the $j-$th neuron in the P-layer and $k-$th neurons in the Q-layer, respectively; $a_j > 0$ and $u_k > 0$ represent the self-inhibitions; $\eta \geq 0$ is constant time delay; $g_k(\cdot)$ and $h_j(\cdot)$ represent the neurons activations; $b_{kj},\ c_{kj},\ v_{jk}$ and w_{jk} indicate the synaptic connection weights of the neurons; G_j and H_k denotes the external inputs of P-Layer and Q-Layer.

The matrix form of the system (5) is represented by:

$$\begin{cases} D^{\lambda}p(t) = -Ap(t)+Bg(q(t))+Cg(q(t-\eta))+G, \\ D^{\lambda}q(t) = -Uq(t)+Vh(p(t))+Wh(p(t-\eta))+H, \end{cases} \tag{6}$$

where $p(t) = (p_1(t),...,p_n(t))^T,\ q(t) = (q_1(t),...,q_m(t))^T,\ A = diag\{a_1,...,a_n\},\ U = diag\{u_1,...,u_m\},$ $B = \big(b_{kj}\big)_{m\times n},\ C = \big(c_{kj}\big)_{m\times n},\ V = \big(v_{jk}\big)_{n\times m},\ W = \big(w_{jk}\big)_{n\times m},\ h(p(\cdot)) = \Big(h_1\big(p_1(\cdot)\big),...,h_n\big(p_n(\cdot)\big)^T\Big)^T,$

$g\big(q(\cdot)\big) = \Big(g_1\big(q_1(\cdot)\big), ..., g_m\big(q_m(\cdot)\big)^T\Big)^T$, $G = \big(G_1, ..., G_n\big)^T$ and $H = \big(H_1, ..., H_m\big)^T$.

Suppose the concerned FBNNs have the following output measurements:

$$\begin{cases} x_p(t) = Ep(t) + R(t, p(t)), \\ x_q(t) = Fq(t) + S(t, q(t)), \end{cases} \tag{7}$$

where $x_p(t) \in \mathbb{R}^n$ and $x_q(t) \in \mathbb{R}^m$ are actual measurement output. $R(t, p(t))$ and $S(t, q(t))$ denote neuron dependent nonlinear disturbances on the network measurement outputs, while $E > 0$ and $F > 0$ are known matrices with constant parameters.

In this letter, the designed full order state estimator of the concerned FBNNs is the following expression:

$$\begin{cases} D^\lambda \hat{p}(t) = -A\hat{p}(t) + Bg(\hat{q}(t)) + Cg(\hat{q}(t-\eta)) + G + M\big[x_p(t) - E\hat{p}(t) + R(t, \hat{p}(t))\big] \\ D^\lambda \hat{q}(t) = -U\hat{q}(t) + Vh(\hat{p}(t)) + Wh(\hat{p}(t-\eta)) + H + N\big[x_q(t) - F\hat{q}(t) + S(t, \hat{q}(t))\big], \end{cases} \tag{8}$$

where $\hat{p}(t) \in \mathbb{R}^n$, $\hat{q}(t) \in \mathbb{R}^m$ are estimation of the neuron states $p(t)$ and $q(t)$, respectively. $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{m\times m}$ are the estimator gain matrices, which is designed to be later.

Let $p\big(t, \sigma^p(t)\big)$, $q\big(t, \sigma^q(t)\big)$ and $\hat{p}\big(t, \varphi^p(t)\big)$, $\hat{q}\big(t, \varphi^q(t)\big)$ be the state evaluations of FBNNs (5) and the state estimator system (8) respectively, with the initial values $\big(p(t), q(t)\big) = \big(\sigma^p(t), \sigma^q(t)\big) \in \mathcal{C}\Big([-\eta, 0], \mathbb{R}^n\Big)$ and $\big(\hat{p}(t), \hat{q}(t)\big) = \big(\varphi^p(t), \varphi^q(t)\big) \in \mathcal{C}\Big([-\eta, 0], \mathbb{R}^m\Big)$.

Denote $z_p(t) = p(t) - \hat{p}(t)$ and $z_q(t) = q(t) - \hat{q}(t)$, the estimator of the error state is described as the following expression:

$$\begin{cases} D^\lambda z_p(t) = -[A + ME]z_p(t) + B\bar{g}(z_q(t)) + C\bar{g}(z_q(t-\eta)) - M\big[\bar{R}(t, z_p(t))\big] \\ D^\lambda z_q(t) = -[U + NF]z_q(t) + V\bar{h}(z_p(t)) + W\bar{h}(z_p(t-\eta)) - N\big[\bar{S}(t, z_q(t))\big], \end{cases} \tag{9}$$

where $\bar{g}(z_q(\cdot)) = g(q(\cdot)) - g(\hat{q}(\cdot))$, $\bar{h}(z_p(\cdot)) = h(p(\cdot)) - h(\hat{p}(\cdot))$, $\bar{R}(t, z_p(t)) = R(t, p(t)) - R(t, \hat{p}(t))$ and $\bar{S}(t, z_q(t)) = R(t, q(t)) - R(t, \hat{q}(t))$.

Let us provide the definition of global Mittag-Leffler state estimator.

Definition 2.8 *The zero solutions of concerned estimator system* (8) *is said to be globally Mittag-Leffler state estimator of the FBNNs* (6), *if the error system* (9) *is globally Mittag-Leffler stable, i.e., for any* $\zeta > 0$, $\varsigma > 0$ *if there exist* $\mathcal{H}\big((\zeta, \varsigma)\big) > 0$, $\nu > 0, \varpi > 0$ *such for any two curves* $\big(p(t)^T, q(t)^T\big)^T$ *and* $\big(\hat{p}(t)^T, \hat{q}(t)^T\big)^T$ *with initial conditions* $\big(\sigma^p(t)^T, \sigma^q(t)^T\big)^T$ *and* $\big(\varphi^p(t)^T, \varphi^q(t)^T\big)^T$, *respectively, as that*

$$\big\|p(t) - \hat{p}(t)\big\| + \big\|q(t) - \hat{q}(t)\big\| \leq \Big\{\mathcal{H}\Big((\zeta, \varsigma)\Big)\mathcal{E}_\lambda\big(-\varpi t^\lambda\big)\Big\}^\nu, \text{ for any } t \geq 0,$$

when $\big\|\sigma^p - \varphi^p\big\| \leq \zeta$, $\big\|\sigma^q - \varphi^q\big\| \leq \varsigma$. Here ϖ is the degree of Mittag-Leffler state estimator, which can be visible the convergence rate as state estimator error tends to zero when time t goes to infinity.

In order to derive the main results, we need the following assumptions and lemmas.

Assumption I. For all $j = 1, 2..., n$, $k = 1, 2, ..., m$, the neuron activation function h_j and g_k satisfy the Lipschitz condition, that is, there exist positive scalars $I_j > 0$ and $J_k > 0$ such that

$$|h_j(p) - h_j(\hat{p})| \leq I_j|p - \hat{p}|, \ |g_k(q) - g_k(\hat{q})| \leq J_k|q - \hat{q}|, \quad \forall\ p,\ \hat{p},\ q,\ \hat{q} \in \mathbb{R}.$$

Assumption II. For all $j = 1, 2..., n,\ k = 1, 2, ..., m,$, the non-linear disturbance functions $R_j(\cdot)$ and $S_k(\cdot)$ are also assumed to Lipschitz continuous:

$$\big|R_j(t,p) - R_j(t,\hat{p})\big| \leq D_j\big|(p-\hat{p})\big|,\ \big|S_k(t,q) - S_k(t,\hat{q})\big| \leq L_k\big|(q-\hat{q})\big|,\ \forall p,\ \hat{p},\ q,\ \hat{q} \in \mathbb{R}.$$

where $D_j,\ L_k$ are known positive constants.

Assumption III. There exist constant D_j such that $|h_j(\cdot)| \leq D_j,\ j = 1, 2, ..., n.$

Assumption IV. There exist constant L_k such that $|g_k(\cdot)| \leq L_k,\ k = 1, 2, ..., m.$

Lemma 2.9 *[53] Let $p(t) \in \mathbb{R}^n$ be a continuous and differentiable vector valued function if there exist a positive definite matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that*

$$D^{\lambda}[p^T(t)\Lambda p(t)] \quad \leq \quad 2p^T(t)\Lambda D^{\lambda}p(t),\ \forall\ \lambda \in (0,1).$$

Lemma 2.10 *[53] For the given positive constant $\alpha > 0$, $p,\ q \in \mathbb{R}^n$ and matrix Υ, then*

$$p^T\Upsilon q \quad \leq \quad \frac{\alpha^{-1}}{2}p^T\Upsilon\Upsilon^T p + \frac{\alpha}{2}q^T q.$$

Lemma 2.11 *[13] For $\vartheta \geq 1$ and if $z_1, ..., z_m \geq 0$, then we have*

$$m^{1-\vartheta}\Big[\sum_{k=1}^{m} z_k\Big]^{\vartheta} \leq \sum_{k=1}^{m} z_k^{\vartheta}.$$

3 Main results

In this section, we present Mittag-Leffler state estimator and Mittag-Leffler adaptive synchronization results.

3.1 Mittag-Leffler state estimator results

The goal of this subsection is to layout the Mittag-Leffler state estimator for FBNNs by using suitable Lyapunov functional and Razumikhin method.

Theorem 3.1 *Under Assumption (I) and (II), let $\Phi_{\min} > \Psi_{\min} > 0$ be two known positive scalar and the enhanced system* (8) *becomes a globally Mittag-Leffler state estimator of system* (6) *if there exist positive definite matrices $\Lambda,\ \Upsilon$, real matrices $Y_p,\ Y_q$ and positive scalars $\alpha_1,\ \alpha_2,\ \delta,\ \lambda_1,\ \lambda_2,\ \mu,\ \xi_1,\ \xi_2,\ \theta_1$ and θ_2 such that the following LMI holds:*

$$\begin{cases} \Phi_1 = -\Lambda A - A^T\Lambda^T - Y_pE - E^TY_p^T + \alpha_1^{-1}\Lambda BB^T\Lambda^T + \alpha_2^{-1}\Lambda CC^T\Lambda^T \\ \qquad +\delta^{-1}\Lambda MM^T\Lambda^T + \delta D^TD + \lambda_1 I^TI < -\xi_1\Lambda \\ \Phi_2 = -\Upsilon U - U^T\Upsilon^T - Y_qF - F^TY_q^T + \lambda_1^{-1}\Upsilon VV^T\Upsilon^T + \lambda_2^{-1}\Upsilon WW^T\Upsilon^T \\ \qquad +\mu^{-1}\Upsilon NN^T\Upsilon^T + \mu L^TL + \alpha_1 J^TJ < -\xi_2\Upsilon, \end{cases} \tag{10}$$

$\Psi_1 = \lambda_2 I^TI < \theta_1\Lambda,\ \Psi_2 = \alpha_2 J^TJ < \theta_2\Upsilon$, where $D = diag\{D_1, ..., D_n\},\ L = diag\{L_1, ..., L_m\},\ I = diag\{I_1, ..., I_n\}$ and $J = diag\{J_1, ..., J_m\}$. Furthermore, the estimator gain matrices M and N are designed by $M = \Lambda^{-1}Y_p$ and $N = \Upsilon^{-1}Y_q$.

Proof. We construct a following Lyapunov functional:

$$V\big(t, z_p(t), z_q(t)\big) = \frac{1}{2}z_p^T(t)\Lambda z_p(t) + \frac{1}{2}z_q^T(t)\Upsilon z_q(t) \tag{11}$$

By virtue of Lemma 2.9, then the fractional derivative of $V(t)$ along the trajectory of error system (9) can be calculated as:

$$\begin{aligned}
D^{\lambda}V\big(t,z_p(t),z_q(t)\big) &\leq z_p^T(t)\Lambda D^{\lambda}[z_p(t)] + z_q^T(t)\Upsilon D^{\lambda}[z_q(t)] \\
&= z_p^T(t)\Lambda\Big[-[A+ME]z_p(t) + B\bar{g}(z_q(t)) + C\bar{g}(z_q(t-\eta)) - M\big[\bar{R}(t,z_p(t))\big]\Big] \\
&\quad + z_q^T(t)\Upsilon\Big[-[U+NF]z_q(t) + V\bar{h}(z_p(t)) + W\bar{h}(z_p(t-\eta)) - N\big[\bar{S}(t,z_q(t))\big]\Big] \\
&\leq \frac{1}{2}z_p^T(t)\big[-\Lambda A - A^T\Lambda^T - Y_pE - E^TY_p^T\big]z^p(t) + z_p^T(t)\Lambda B\bar{g}(z_q(t)) + z_p^T(t)\Lambda \\
&\quad \times C\bar{g}(z_q(t-\eta)) - z_p^T(t)\Lambda M\bar{R}(t,z_p(t)) + \frac{1}{2}z_q^T(t)\big[-\Upsilon U - U^T\Upsilon^T - Y_qF - F^T \\
&\quad \times Y_q^T\big]z_q(t) + z_q^T(t)\Upsilon V\bar{h}(z_p(t)) + z_q^T(t)\Upsilon W\bar{h}(z_p(t-\eta)) - z_q^T(t)\Upsilon N\bar{S}(t,z_q(t))
\end{aligned}$$

By means of Assumption I, Assumption II and Lemma 2.10, we get

$$\begin{aligned}
D^{\lambda}V\big(t,z_p(t),z_q(t)\big) &\leq \frac{1}{2}z_p^T(t)\Big[-\Lambda A - A^T\Lambda^T - Y_pE - E^TY_p^T + \alpha_1^{-1}\Lambda BB^T\Lambda^T + \alpha_2^{-1}\Lambda CC^T\Lambda^T \\
&\quad + \delta^{-1}\Lambda MM^T\Lambda^T + \delta D^TD + \lambda_1 I^TI\Big]z_p(t) + \frac{1}{2}z_p^T(t-\eta)\Big[\lambda_2 I^TI\Big]z_p(t-\eta) \\
&\quad + \frac{1}{2}z_q^T(t)\big[-\Upsilon U - U^T\Upsilon^T - Y_qF - F^TY_q^T + \lambda_1^{-1}\Upsilon VV^T\Upsilon^T + \lambda_2^{-1}\Upsilon WW^T\Upsilon^T \\
&\quad + \mu^{-1}\Upsilon NN^T\Upsilon^T + \mu L^TL + \alpha_1 J^TJ\Big]z_q(t) + \frac{1}{2}z_q^T(t-\eta)\Big[\alpha_2 J^TJ\Big]z_q(t-\eta) \\
&\leq -\Phi_{\min}V\big(t,z_p(t),z_q(t)\big) + \Psi_{\max}\sup_{t-\eta\leq\omega\leq t}V\big(\omega,z_p(\omega),z_q(\omega)\big)
\end{aligned}$$

where $\Phi_1 < -\xi_1\Lambda,\ \Phi_2 < -\xi_2\Upsilon,\ \Psi_1 < \theta_1\Lambda,\ \Psi_2 < \theta_2\Upsilon,\ \Phi_{\min} = \min\{\xi_1,\xi_2\}$ and $\Psi_{\max} = \max\{\theta_1,\theta_2\}$. Based on the above estimate, any solution of the system (9) satisfies the following Razumikhin condition [15].

$$V\big(\omega,z_p(\omega),z_q(\omega)\big) \leq V\big(t,z_p(t),z_q(t)\big),\ t-\eta\leq\omega\leq t.$$

That is

$$D^{\lambda}V\big(t,z_p(t),z_q(t)\big) \leq -\big[\Phi_{\min} - \Psi_{\max}\big]V\big(t,z_p(t),z_q(t)\big).$$

Then by aid of Lemma 2.6, it follows that

$$V\big(t,z_p(t),z_q(t)\big) \leq \big[V\big(0,z_p(0),z_q(0)\big)\big]\mathcal{E}_{\lambda}\Big[\big(\Psi_{\max} - \Phi_{\min}\big)t^{\lambda}\Big],\ \forall\, t\in[0,+\infty). \tag{12}$$

Otherwise, Lyapunov function $V\big(t,z_p(t),z_q(t)\big)$ satisfies

$$\frac{1}{2}\tilde{\lambda}_{\min}(\Lambda)\|z_p(t)\|^2 + \frac{1}{2}\tilde{\lambda}_{\min}(\Upsilon)\|z_q(t)\|^2 \leq V\big(t,z_p(t),z_q(t)\big) \leq \frac{1}{2}\tilde{\lambda}_{\max}(\Lambda)\|z_p(t)\|^2 + \frac{1}{2}\tilde{\lambda}_{\max}(\Upsilon)\|z_q(t)\|^2,$$

which implies

$$\frac{1}{2}\epsilon_{\min}\big[\|z_p(t)\|^2 + \|z_q(t)\|^2\big] \leq V\big(t,z_p(t),z_q(t)\big) \leq \frac{1}{2}\epsilon_{\max}\big[\|z_p(t)\|^2 + \|z_q(t)\|^2\big] \tag{13}$$

where $\epsilon_{\min} = \min\{\tilde{\lambda}_{\min}(\Lambda),\tilde{\lambda}_{\min}(\Upsilon)\}$ and $\epsilon_{\max} = \max\{\tilde{\lambda}_{\max}(\Lambda),\tilde{\lambda}_{\max}(\Upsilon)\}$.

On the other hand,

$$
\begin{aligned}
V\big(0, z_p(0), z_q(0)\big) &\leq \frac{1}{2}\tilde{\lambda}_{\max}(\Lambda)\|\sigma_p - \varphi_p\|^2 + \frac{1}{2}\tilde{\lambda}_{\max}(\Upsilon)\|\sigma_q - \varphi_q\|^2, \\
&\leq \frac{1}{2}\epsilon_{\max}\big[\|\sigma_p - \varphi_p\|^2 + \|\sigma_q - \varphi_q\|^2\big] \qquad (14)
\end{aligned}
$$

Combining Eq.(12), Eq.(13) and Eq.(14), it follows that

$$
\big[\|z_p(t)\|^2 + \|z_q(t)\|^2\big] \leq \varpi\big[\|\sigma_p - \varphi_p\|^2 + \|\sigma_q - \varphi_q\|^2\big]\mathcal{E}_\lambda\Big[\big(\Psi_{\max} - \Phi_{\min}\big)t^\lambda\Big], \ \forall\, t \in [0, +\infty). \qquad (15)
$$

By utilizing Lemma 2.11, it follows that

$$
\big[\|p(t) - \hat{p}(t)\| + \|q(t) - \hat{q}(t)\|\big] \leq \Big[\mathcal{H}\big((\zeta, \varsigma)\big)\mathcal{E}_\lambda\big[\big(\Psi_{\max} - \Phi_{\min}\big)t^\lambda\big]\Big]^{\frac{1}{2}}, \ \forall\, t \in [0, +\infty).
$$

where $\varpi = \frac{\epsilon_{\max}}{\epsilon_{\min}}$ and $\mathcal{H}\big((\zeta, \varsigma)\big) = 2\varpi(\zeta + \varsigma) \geq 0$. According to Definition 2.8, the system (8) becomes a globally Mittag-Leffler state estimator of the system (6). Hence completed the proof.

Corollary 3.2 *Under Assumption (I) and (II), the enhanced system* (8) *without delay term becomes a globally Mittag-Leffler state estimator of system* (6) *without delay term if there exist positive definite matrices* $\Lambda,\ \Upsilon$, *real matrices* $Y_p,\ Y_q$ *and positive scalars* $\alpha_1,\ \delta,\ \xi_3,\ \xi_4,\ \lambda_1$ *and* μ *such that the following LMI holds:*

$$
\begin{cases}
\Phi_3 = -\Lambda A - A^T\Lambda^T - Y_pE - E^TY_p^T + \alpha_1^{-1}\Lambda BB^T\Lambda^T + \delta^{-1}\Lambda MM^T\Lambda^T + \delta D^TD + \lambda_1 I^TI < -\xi_3\Lambda \\
\Phi_4 = -\Upsilon U - U^T\Upsilon^T - Y_qF - F^TY_q^T + \lambda_1^{-1}\Upsilon VV^T\Upsilon^T + \mu^{-1}\Upsilon NN^T\Upsilon^T + \mu L^TL + \alpha_1 J^TJ < -\xi_4\Upsilon,
\end{cases}
$$

where $D = diag\{D_1, ..., D_n\},\ L = diag\{L_1, ..., L_m\},\ I = diag\{I_1, ..., I_n\}$ *and* $J = diag\{J_1, ..., J_m\}$. *Furthermore, the estimator gain matrices* M *and* N *are designed by* $M = \Lambda^{-1}Y_p$ *and* $N = \Upsilon^{-1}Y_q$.

Remark 3.3 *The authors in [4] considered a class of state estimation of fractional order neural networks without time delay via absolute value Lyapunov functional and Mittag-Leffler functions. In [44], state estimation of fractional-order memristor-based neural networks with time delays were investigated by utilizing LMI techniques and positive definite quadratic Lyapunov functional. However, the Mittag-Leffler state estimator design for FBNNs with (or without) time delay has not been seen in the previous literature. Therefore, the results based on the state estimation in this paper are completely new in contrast with the existing works.*

Remark 3.4 *Theorem 3.1 and Corollary 3.2 are derived by means of LMI techniques. To find the feasible solution of the LMI in the case of bigger LMIs in size, can be get solved by the interior point algorithms in convex optimization technique and the LMI toolbox in MATLAB. Yet there is a increase in Computational time.*

3.2 Mittag-Leffler synchronization results

In this subsection, sufficient conditions are given to realize the global Mittag-Leffler synchronization for FBNNs via adaptive feedback controller.

Model (5) is called as a master system. By means of master-slave synchronization concept, the corresponding slave system of Model (5) can be expressed in the following form:

$$
\begin{cases}
D^\lambda\tilde{p}_j(t) = -a_j\tilde{p}_j(t) + \sum_{k=1}^m b_{kj}g_k(\tilde{q}_k(t)) + \sum_{k=1}^m c_{kj}g_k(\tilde{q}_k(t-\eta)) + G_j + \delta_j(t), \\
D^\lambda\tilde{q}_k(t) = -u_k\tilde{q}_k(t) + \sum_{j=1}^n v_{jk}h_j(\tilde{p}_j(t)) + \sum_{j=1}^n w_{jk}h_j(\tilde{p}_j(t-\eta)) + H_k + \theta_k(t),
\end{cases} \qquad (16)
$$

where $\tilde{p}_j(t)$ and $\tilde{q}_k(t)$ are corresponding state variable of the slave system, while $\delta_j(t)$ and $\theta_k(t)$ are adaptive delayed feedback controller and all other parameters are the similar as those in master system.

Let $p\big(t, \sigma^p(t)\big),\ q\big(t, \sigma^q(t)\big)$ and $\tilde{p}\big(t, \gamma^p(t)\big),\ \tilde{q}\big(t, \gamma^q(t)\big)$ be the state trajectories of the master system (5) and the slave system (16) respectively, with the initial values $\big(p(t), q(t)\big) = \big(\sigma^p(t), \sigma^q(t)\big) \in \mathcal{C}\Big([-\eta, 0], \mathbb{R}^n\Big)$ and $\big(\tilde{p}(t), \tilde{q}(t)\big) = \big(\gamma^p(t), \gamma^q(t)\big) \in \mathcal{C}\Big([-\eta, 0], \mathbb{R}^m\Big)$.

Let $e_{pj}(t) = \tilde{p}_j(t) - p_j(t),\ e_{qk}(t) = \tilde{q}_k(t) - q_k(t),\ j = 1, 2, ..., n,\ k = 1, 2, ...m$. Then the fractional order synchronization error system is defined as:

$$\begin{cases} D^\lambda e_{pj}(t) = -a_j e_{pj}(t) + \tilde{E}_j(t) + \delta_j(t), \\ D^\lambda e_{qk}(t) = -u_k e_{qk}(t) + \tilde{F}_k(t) + \theta_k(t), \end{cases} \tag{17}$$

where

$$\begin{aligned} \tilde{E}_j(t) &= \sum_{k=1}^{m} b_{kj} g_k(\tilde{q}_k(t)) - \sum_{k=1}^{m} b_{kj} g_k(q_k(t)) + \sum_{k=1}^{m} c_{kj} g_k(\tilde{q}_k(t-\eta)) - \sum_{k=1}^{m} c_{kj} g_k(q_k(t-\eta)) \\ \tilde{F}_k(t) &= \sum_{j=1}^{n} v_{jk} h_j(\tilde{p}_j(t)) - \sum_{j=1}^{n} v_{jk} h_j(p_j(t)) + \sum_{j=1}^{n} w_{jk} h_j(\tilde{p}_j(t-\eta)) - \sum_{j=1}^{n} w_{jk} h_j(p_j(t-\eta)). \end{aligned}$$

Lemma 3.5 $|\tilde{E}_j(t)| \leq \beta_j^p$, *where* $\beta_j^p = \sum_{k=1}^{m} 2L_k\big(|b_{kj}| + |c_{kj}|\big)$ *for* $j = 1, 2..., n$.
Proof. *With the aid of Assumption III, it can be easily proved from the definition of* $|\tilde{E}_j(t)|$.

Lemma 3.6 $|\tilde{F}_k(t)| \leq \beta_k^q$, *where* $\beta_k^q = \sum_{j=1}^{n} 2D_j\big(|v_{jk}| + |w_{jk}|\big)$ *for* $k = 1, 2..., m$.
Proof. *With the aid of Assumption IV, it can be easily proved from the definition of* $|\tilde{F}_k(t)|$.

The fractional order synchronization error system (17) can also be converted into the following matrix form:

$$\begin{cases} D^\lambda e_p(t) = -A e_p(t) + \tilde{E}(t) + \delta(t), \\ D^\lambda e_q(t) = -U e_q(t) + \tilde{F}(t) + \theta(t), \end{cases} \tag{18}$$

where $e_p(t) = \big(e_{p1}(t), ..., e_{pn}(t)\big)^T,\ e_q(t) = \big(e_{q1}(t), ..., e_{qm}(t)\big)^T,\ \tilde{E}(t) = \big(\tilde{E}_1(t), ..., \tilde{E}_n(t)\big)^T,\ \tilde{F}(t) = \big(\tilde{F}_1(t), ..., \tilde{F}_m(t)\big)^T,\ \delta(t) = \big(\delta_1(t), ...\delta_m(t)\big)^T$ and $\theta(t) = \big(\theta_1(t), ...\theta_m(t)\big)^T$.

Designing the adaptive feedback control for fractional order system (16) which involves the sign function in the following expression:

$$\begin{cases} \delta(t) = -\alpha(t) e_p(t) - \mu(t)\, \mathrm{sgn}[e_p(t)], \\ \theta(t) = -\phi(t) e_q(t) - \xi(t)\, \mathrm{sgn}[e_q(t)] \end{cases} \tag{19}$$

with the adaptive updated law is

$$\begin{cases} D^\lambda \alpha_j(t) = \sum_{k=1}^{m} e_{pk}(t) \pi_j e_{pj}(t),\ D^\lambda \mu_j(t) = \rho_j |e_{pj}(t)| \\ D^\lambda \phi_k(t) = \sum_{j=1}^{n} e_{qj}(t) \varepsilon_k e_{qk}(t),\ D^\lambda \mu_k(t) = \varrho_k |e_{qk}(t)|, \end{cases}$$

where $\alpha(t) = diag\{\alpha_1(t), ..., \alpha_n(t)\},\ \mu(t) = diag\{\mu_1(t), ..., \mu_n(t)\},\ \phi(t) = diag\{\phi_1(t), ..., \phi_m(t)\}$ and $\xi(t) = diag\{\xi_1(t), ..., \xi_m(t)\}$. $\pi_j,\ \rho_j,\ \varepsilon_k$ and ϱ_k are positive scalars for $j = 1, 2, ..., n$ and $k = 1, 2, ..., m$.

Finally, we will provide the definition of global Mittag-Leffler synchronization, which will be used to the main part of theorem.

Definition 3.7 *The slave system* (5) *is said to be global Mittag-Leffler synchronized with in the master system* (16) *based on the controller* (19) *for any* $\zeta_1 > 0,\ \varsigma_1 > 0$ *if there exist* $\mathcal{H}\big((\zeta_1, \varsigma_1)\big) > 0,\ \nu > 0, \varpi > 0$ *and* $\mathcal{T} > 0$ *such for any two curves* $\big(p(t)^T, q(t)^T\big)^T$ *and* $\big(\tilde{p}(t)^T, \tilde{q}(t)^T\big)^T$ *with initial conditions* $\big(\sigma^p(t)^T, \sigma^q(t)^T\big)^T$ *and* $\big(\gamma^p(t)^T, \gamma^q(t)^T\big)^T$*, respectively, as that*

$$\big\|\tilde{p}(t) - p(t)\big\| + \big\|\tilde{q}(t) - q(t)\big\| \leq \Big\{\mathcal{H}\Big((\zeta_1, \varsigma_1)\Big)\mathcal{E}_\lambda\big(-\varpi t^\lambda\big)\Big\}^\nu, \ \textit{for any}\ t \geq \mathcal{T},$$

when $\big\|\gamma^p - \sigma^p\big\| \leq \zeta_1,\ \big\|\gamma^q - \sigma^q\big\| \leq \varsigma_1$. Here ϖ is the degree of Mittag-Leffler synchronization, which can be visible the convergence rate as synchronization error goes to zero when time t approaches to infinity.

Theorem 3.8 *Suppose the assumption* (*III*) *and* (*IV*) *hold, then the slave system* (16) *is globally Mittag-Leffler synchronized with the master system* (5) *under the controller* (19) *if there exist positive definite matrices* $\tilde{\Lambda} = diag\big\{\tilde{\Lambda}_1, ..., \tilde{\Lambda}_n\big\} > 0,\ \tilde{\Upsilon} = diag\big\{\tilde{\Upsilon}_1, ..., \tilde{\Upsilon}_m\big\} > 0$, *and positive scalars* r_1 *and* r_2 *such that the following LMI holds:*

$$\begin{cases}\Omega_1 = -\Lambda A - A^T\tilde{\Lambda}^T - \tilde{\Lambda}\Pi - \Pi^T\Lambda^T < -r_1\tilde{\Lambda}, \\ \Omega_2 = -\Upsilon U - U^T\tilde{\Upsilon}^T - \tilde{\Upsilon}\Phi - \Phi^T\tilde{\Upsilon}^T < -r_2\tilde{\Upsilon},\end{cases} \tag{20}$$

Furthermore, the control gains are subjected to

$$\mu_j \geq \beta_j^p = \sum_{k=1}^{m} 2L_k\big(|b_{kj}| + |c_{kj}|\big),\ \xi_k \geq \beta_k^q = \sum_{j=1}^{n} 2D_j\big(|v_{jk}| + |w_{jk}|\big) \tag{21}$$

where $\Pi = diag\{\alpha_1, ..., \alpha_n\} > 0$ *and* $\Phi = diag\{\phi_1, ..., \phi_m\} > 0$.

Proof. We construct a following Lyapunov functional:

$$\tilde{V}\big(t, e_p(t), e_q(t)\big) = \tilde{U}(t) + \tilde{W}(t) \tag{22}$$

where

$$\tilde{U}(t) = \frac{1}{2}e_p^T(t)\tilde{\Lambda}e_p(t) + \frac{1}{2}e_q^T(t)\tilde{\Upsilon}e_q(t),$$

$$\tilde{W}(t) = \sum_{j=1}^{n}\left[\frac{\tilde{\Lambda}_j}{2\pi_j}\big[\alpha_j(t) - \alpha_j\big]^2 + \frac{\tilde{\Lambda}_j}{2\rho_j}\big[\mu_j(t) - \mu_j\big]^2\right] + \sum_{k=1}^{m}\left[\frac{\tilde{\Upsilon}_k}{2\varepsilon_k}\big[\phi_k(t) - \phi_k\big]^2 + \frac{\tilde{\Upsilon}_k}{2\varrho_k}\big[\xi_k(t) - \xi_k\big]^2\right]$$

where $\alpha_j,\ \mu_j,\ \phi_k$ and ξ_k are positive constants.

By means of Lemma 2.9, then the fractional-order derivative of Lyapunov functional $\tilde{U}(t)$ along the trajectory of error system (18) can be calculated as:

$$\begin{aligned} D^\lambda\tilde{U}(t) &\leq e_p^T(t)\tilde{\Lambda}D^\lambda[e_p(t)] + e_q^T(t)\tilde{\Upsilon}D^\lambda[e_q(t)] \\ &= e_p^T(t)\tilde{\Lambda}\Big[-\big(A + \alpha(t)\big)e_p(t) + \tilde{E}(t) - \mu(t)\psi_p(t)\Big] \\ &\quad + e_q^T(t)\tilde{\Upsilon}\Big[-\big(U + \phi(t)\big)e_q(t) + \tilde{F}(t) - \xi(t)\psi_q(t)\Big] \end{aligned} \tag{23}$$

where $\psi_p(t) = \big)^T\psi_{p1}(t), ..., \psi_{pn}(t)\big)^T,\ \psi_q(t) = \big(\psi_{q1}(t), ..., \psi_{qm}(t)\big)^T$ implies that, $\psi_{pj}(t) = \mathrm{sgn}(e_{pj}(t))$ if $e_{pj}(t) \neq 0$ and $\psi_{qk}(t) = \mathrm{sgn}(e_{qk}(t))$ if $e_{qk}(t) \neq 0$, once $e_{pj}(t) = 0,\ e_{qk}(t) = 0$, while $\psi_{pj}(t)$ and $\psi_{qk}(t)$ can be chosen on $[-1, 1]$.

Then, the last few term of right hand side of Eq.(23) can be given as follows:

$$
\begin{aligned}
e_p^T(t)\tilde{\Lambda}\big[\tilde{E}(t)-\mu(t)\psi_p(t)\big] &\leq \sum_{j=1}^{n}\tilde{\Lambda}_j|e_{pj}(t)|\Bigg[\beta_j^p-\mu_j(t)\psi_{pj}(t)\Bigg] \\
&\leq \sum_{j=1}^{n}\tilde{\Lambda}_j\beta_j^p|e_{pj}(t)|-\sum_{j=1}^{n}\tilde{\Lambda}_j\mu_j(t)|e_{pj}(t)| \qquad (24)
\end{aligned}
$$

and

$$
\begin{aligned}
e_q^T(t)\tilde{\Upsilon}\big[\tilde{F}(t)-\xi(t)\psi_q(t)\big] &\leq \sum_{k=1}^{m}\tilde{\Upsilon}_k|e_{qk}(t)|\Bigg[\beta_k^q-\xi_k(t)\psi_{qk}(t)\Bigg] \\
&\leq \sum_{k=1}^{m}\tilde{\Upsilon}_k\beta_k^q|e_{qk}(t)|-\sum_{k=1}^{m}\tilde{\Upsilon}_k\xi_k(t)|e_{qk}(t)|, \qquad (25)
\end{aligned}
$$

where Lemma 3.5 and Lemma 3.6 has been used.

Next, by calculating the Caputo derivative of $\tilde{W}(t)$, one can read that

$$
\begin{aligned}
D^\lambda\tilde{W}(t) &\leq \sum_{j=1}^{n}\Bigg[\frac{\tilde{\Lambda}_j}{\pi_j}\big[\alpha_j(t)-\alpha_j\big]D^\lambda[\alpha_j(t)]+\frac{\tilde{\Lambda}_j}{\rho_j}\big[\mu_j(t)-\mu_j\big]D^\lambda[\mu_j(t)]\Bigg] \\
&\quad+\sum_{k=1}^{m}\Bigg[\frac{\tilde{\Upsilon}_k}{\varepsilon_k}\big[\phi_k(t)-\phi_k\big]D^\lambda[\phi_k(t)]+\frac{\tilde{\Upsilon}_k}{\varrho_j}\big[\xi_k(t)-\xi_k\big]D^\lambda[\xi_k(t)]\Bigg] \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m}e_{pk}(t)\tilde{\Lambda}_j\big[\alpha_j(t)-\alpha_j\big]e_{pj}(t)-\sum_{j=1}^{n}\tilde{\Lambda}_j\mu_j|e_{pj}(t)|+\sum_{j=1}^{n}\tilde{\Lambda}_j\mu_j(t)|e_{pj}(t)| \\
&\quad+\sum_{k=1}^{m}\sum_{j=1}^{n}e_{qj}(t)\tilde{\Upsilon}_k\big[\phi_k(t)-\phi_k\big]e_{qk}(t)+\sum_{k=1}^{m}\tilde{\Upsilon}_k\xi_k(t)|e_{qk}(t)|-\sum_{k=1}^{m}\tilde{\Upsilon}_k\xi_k|e_{qk}(t)| \\
&= e_p^T(t)\tilde{\Lambda}\Big[\alpha(t)-\Pi\Big]e_p(t)-\sum_{j=1}^{n}\tilde{\Lambda}_j\mu_j|e_{pj}(t)|+\sum_{j=1}^{n}\tilde{\Lambda}_j\mu_j(t)|e_{pj}(t)| \\
&\quad+e_q^T(t)\tilde{\Upsilon}\Big[\phi(t)-\Phi\Big]e_q(t)+\sum_{k=1}^{m}\tilde{\Upsilon}_k\xi_k(t)|e_{qk}(t)|-\sum_{k=1}^{m}\tilde{\Upsilon}_k\xi_k|e_{qk}(t)| \qquad (26)
\end{aligned}
$$

Combining Eq.(23)-Eq.(25) and Eq.(26), one has obtain

$$
\begin{aligned}
D^\lambda\tilde{V}\Big(t,e_p(t),e_q(t)\Big) &\leq e_p^T(t)\big[-\tilde{\Lambda}A-\tilde{\Lambda}\Pi\Big]e_p(t)+e_q^T(t)\Big[-\tilde{\Upsilon}U-\tilde{\Upsilon}\Phi\Big]e_q(t)-\sum_{j=1}^{n}\tilde{\Lambda}_j\mu_j|e_{pj}(t)| \\
&\quad+\sum_{j=1}^{n}\tilde{\Lambda}_j\beta_j^p|e_{pj}(t)|-\sum_{k=1}^{m}\tilde{\Upsilon}_k\xi_k|e_{qk}(t)|+\sum_{k=1}^{m}\tilde{\Upsilon}_k\beta_k^q|e_{qk}(t)| \\
&\leq \frac{1}{2}e_p^T(t)\big[-\tilde{\Lambda}A-A^T\tilde{\Lambda}^T-\tilde{\Lambda}\Pi-\Pi^T\tilde{\Lambda}^T\big]+\frac{1}{2}e_q^T(t)\Big[-\tilde{\Upsilon}U-U^T\tilde{\Upsilon}^T \\
&\quad-\tilde{\Upsilon}\Phi-\Phi^T\tilde{\Upsilon}^T\Big]-\sum_{j=1}^{n}\tilde{\Lambda}_j\Big[\mu_j-\beta_j^p\Big]|e_{pj}(t)|-\sum_{k=1}^{m}\tilde{\Upsilon}_k\Big[\xi_k-\beta_k^q\Big]|e_{qk}(t)| \\
&\leq -\frac{1}{2}e_p^T(t)r_1\tilde{\Lambda}e_p(t)-\frac{1}{2}e_q^T(t)r_2\tilde{\Upsilon}e_q(t) \\
&\leq -\Omega_{\min}\tilde{V}\Big(t,e_p(t),e_q(t)\Big), \qquad (27)
\end{aligned}
$$

where $\Omega_{\min} = \min\{r_1,\ r_2\}$. Then by means of Lemma 2.7, it follows that

$$\tilde{U}(t) \leq \big[V\big(0, e_p(0), e_q(0)\big)\big]\mathcal{E}_\lambda\Big[-\Omega_{\min}t^\lambda\Big],\ \forall\ t \geq \mathcal{T}. \tag{28}$$

Otherwise, $\tilde{U}(t)$ satisfies

$$\frac{1}{2}\tilde{\epsilon}_{\min}\big[\|e_p(t)\|^2 + \|e_q(t)\|^2\big] \leq \tilde{U}(t) \leq \frac{1}{2}\tilde{\epsilon}_{\max}\big[\|e_p(t)\|^2 + \|e_q(t)\|^2\big] \tag{29}$$

where $\tilde{\epsilon}_{\min} = \min\{\tilde{\lambda}_{\min}(\tilde{\Lambda}), \tilde{\lambda}_{\min}(\tilde{\Upsilon})\}$ and $\tilde{\epsilon}_{\max} = \max\{\tilde{\lambda}_{\max}(\tilde{\Lambda}), \tilde{\lambda}_{\max}(\tilde{\Upsilon})\}$. On the other hand,

$$\begin{aligned} V\big(0, e_p(0), e_q(0)\big) &\leq \frac{1}{2}\tilde{\lambda}_{\max}(\tilde{\Lambda})\|\gamma_p - \varphi_p\|^2 + \frac{1}{2}\tilde{\lambda}_{\max}(\tilde{\Upsilon})\|\gamma_q - \varphi_q\|^2 + \frac{1}{2}\sum_{j=1}^{n}\Bigg[\frac{\tilde{\Lambda}_j}{\pi_j}\big[\alpha_j(0) - \alpha_j\big]^2 \\ &\quad + \frac{\tilde{\Lambda}_j}{\rho_j}\big[\mu_j(0) - \mu_j\big]^2\Bigg] + \sum_{k=1}^{m}\Bigg[\frac{\tilde{\Upsilon}_k}{\varepsilon_k}\big[\phi_k(0) - \phi_k\big]^2 + \frac{\tilde{\Upsilon}_k}{\varrho_k}\big[\xi_k(0) - \xi_k\big]^2\Bigg], \\ &\leq \frac{1}{2}\Bigg[\tilde{\epsilon}_{\max}\Big[\|\gamma_p - \varphi_p\|^2 + \|\gamma_q - \varphi_q\|^2\Big] + \sum_{j=1}^{n}\frac{\tilde{\Lambda}_j}{\pi_j}\big[\alpha_j(0) - \alpha_j\big]^2 \\ &\quad + \sum_{j=1}^{n}\frac{\tilde{\Lambda}_j}{\rho_j}\big[\mu_j(0) - \mu_j\big]^2 + \sum_{k=1}^{m}\frac{\tilde{\Upsilon}_k}{\varepsilon_k}\big[\phi_k(0) - \phi_k\big]^2 + \sum_{k=1}^{m}\frac{\tilde{\Upsilon}_k}{\varrho_k}\big[\xi_k(0) - \xi_k\big]^2\Bigg], \end{aligned} \tag{30}$$

According to Eq.(28)-Eq.(30) and by utilizing Lemma 2.11, it follows that

$$\Big[\|\tilde{p}(t) - p(t)\| + \|\tilde{q}(t) - q(t)\|\Big] \leq \Big[\mathcal{H}\big((\zeta_1, \varsigma_1)\big)\mathcal{E}_\lambda\big[-\Omega_{\min}\,t^\lambda\big]\Big]^{\frac{1}{2}},\ t \geq \mathcal{T}.$$

where

$$\begin{aligned} \mathcal{H}\big((\zeta, \varsigma)\big) &= \frac{2}{\tilde{\epsilon}_{\min}}\Bigg[\tilde{\epsilon}_{\max}\zeta_1 + \tilde{\epsilon}_{\max}\varsigma_1 + \sum_{j=1}^{n}\frac{\tilde{\Lambda}_j}{\pi_j}\big[\alpha_j(0) - \alpha_j\big]^2 + \sum_{j=1}^{n}\frac{\tilde{\Lambda}_j}{\rho_j}\big[\mu_j(0) - \mu_j\big]^2 \\ &\quad + \sum_{k=1}^{m}\frac{\tilde{\Upsilon}_k}{\varepsilon_k}\big[\phi_k(0) - \phi_k\big]^2 + \sum_{k=1}^{m}\frac{\tilde{\Upsilon}_k}{\varrho_k}\big[\xi_k(0) - \xi_k\big]^2\Bigg] \geq 0. \end{aligned}$$

By virtue of Definition 3.7, the slave system (16) is globally Mittag-Leffler synchronized with the master system (5) under the controller (19). Hence completed the proof.

When $C = W = 0$, i.e., the system (5) and (16) has without time delays. Then the following corollary is directly obtained from Theorem 3.8.

Corollary 3.9 *Suppose the assumption* (III) *and* (IV) *hold, then the slave system* (16) *is globally Mittag-Leffler synchronized with the master system* (5) *under the controller* (19) *if there exist positive definite matrices* $\tilde{\Lambda} = diag\{\tilde{\Lambda}_1, ..., \tilde{\Lambda}_n\} > 0$, $\tilde{\Upsilon} = diag\{\tilde{\Upsilon}_1, ..., \tilde{\Upsilon}_m\} > 0$, *and positive scalars* r_1 *and* r_2 *such that the following LMI holds:*

$$\begin{cases} \Omega_1 = -\Lambda A - A^T\tilde{\Lambda}^T - \tilde{\Lambda}\Pi - \Pi^T\Lambda^T < -r_1\tilde{\Lambda}, \\ \Omega_2 = -\Upsilon U - U^T\tilde{\Upsilon}^T - \tilde{\Upsilon}\Phi - \Phi^T\tilde{\Upsilon}^T < -r_2\tilde{\Upsilon}, \end{cases} \tag{31}$$

Furthermore, the control gains are subjected to

$$\mu_j \geq \sum_{k=1}^{m} 2L_k|b_{kj}|,\ \xi_k \geq \sum_{j=1}^{n} 2D_j|v_{jk}|, \tag{32}$$

where $\Pi = diag\{\alpha_1, ..., \alpha_n\} > 0$ *and* $\Phi = diag\{\phi_1, ..., \phi_m\} > 0$.

Remark 3.10 *Reviewing the existing works in the literatures, a large number of studies on the adaptive synchronization of fractional order neural networks can be found by many researchers, see Ref [3, 6, 38, 46]. But there is few results focused on adaptive synchronization of fractional order BAM neural networks with time delays [33] and by using maximum absolute value method. However, there is no one studied the adaptive synchronization of fractional order BAM neural networks with (or without) time delays via positive definite quadratic Lyapunov-functional and Mittag-Leffler functions. Up to best of authors knowledge, the proposed synchronization results are considered as new.*

Remark 3.11 *In some earlier literature, the Lyapunov-functionals are constructed by nonsmooth absolute functions with one-norm, two-norm and p-norm* $(p > 1)$ *functions and the obtained results are all in the form of matrix elements, which caused greater complicated calculation for the system solutions because their results has to check one by one for n times. To overcome the above computational burden in Theorem 3.1 and 3.8, the Lyapunov functions are used in positive definite quadratic function and the results can be checked easily by LMI MATLAB toolbox. Different from the synchronization results in [3, 6, 34, 48, 52, 54], the activations are only assumed to be bounded, which helps to reduce the computational complexity for computation. Furthermore, in our proposed synchronization criteria, we used less number of whole matrices and matrix elements which is much more simpler, it leads to less conservatism.*

Remark 3.12 *Different from the control techniques in the earlier publications [9, 48, 51], adaptive feedback control* (19) *is more effective and the designer requirements of adaptive feedback controllers are very effortless. The sufficiently small control gains* π_j, ρ_j, ε_k *and* ϱ_k *of* (19) *would lead to small control inputs. However the required synchronization speed may be very gradual. Subsequently, when the adaptive feedback control used to realize synchronization goal, the adaptive control gains ought to be chosen to increase the synchronization speed and to reduce the values of control inputs.*

4 Numerical Simulations

In this section, two numerical examples and simulations are given to reveal the effectiveness of the theoretical results derived formerly.

Example 4.1 *Consider the three dimensional FBNNs*

$$\begin{cases} D^{\lambda}p(t) = -Ap(t) + Bg(q(t)) + Cg(q(t-\eta)) + G, \\ D^{\lambda}q(t) = -Uq(t) + Vh(p(t)) + Wh(p(t-\eta)) + H, \end{cases}$$

with network measurement output is

$$\begin{cases} x_p(t) = Ep(t) + R(t, p(t)), \\ x_q(t) = Fq(t) + S(t, q(t)), \end{cases}$$

where $\lambda = 0.98,\ p(t) = \big(p_1(t), p_2(t), p_3(t)\big)^T,\ q(t) = \big(q_1(t), q_2(t), q_3(t)\big)^T,\ g(q(t)) = \tanh(q(t)),\ S(t, q(t)) = \sin(q(t)),\ h(p(t)) = \tanh(p(t)),\ R(t, p(t)) = \sin(p(t)),\ G = H = [0, 0, 0]^T,\ A = diag\{3,\ 3,\ 3\},\ U = diag\{2.75,\ 2.75,\ 2.75\},\ E = diag\{1,\ 1.2,\ 0.8\},\ F = diag\{0.5,\ 0.5,\ 0.5\},\ \eta = 1.5,\ \xi_1 = 1.5,\ \xi_2 = 2,\ \theta_1 = 1,\ \theta_2 = 1.3,$

$$\begin{aligned} B &= \begin{bmatrix} 2.8 & 1.2 & -1.1 \\ -2.2 & 2.3 & 2.2 \\ 1.2 & -1.1 & 2.1 \end{bmatrix}, \quad C = \begin{bmatrix} 2.8 & 1 & -2.1 \\ -2 & 2.3 & 0.2 \\ 1.5 & -3 & 2.1 \end{bmatrix}, \\ V &= \begin{bmatrix} 2.5 & 1.2 & -2.2 \\ -2.6 & 2.3 & 3 \\ 1.4 & -1.1 & 1.8 \end{bmatrix}, \quad W = \begin{bmatrix} 0.5 & 1.5 & -2.1 \\ -1.3 & 2.3 & 1.4 \\ 1 & -3 & 1.8 \end{bmatrix}, \end{aligned}$$

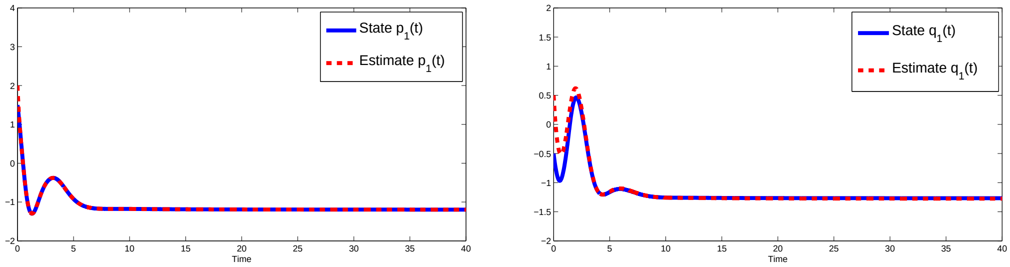

Figure 1: State evolution of $p_1(t)$, $\hat{p}_1(t)$, $q_1(t)$, $\hat{q}_1(t)$ and its estimations with Example 4.1

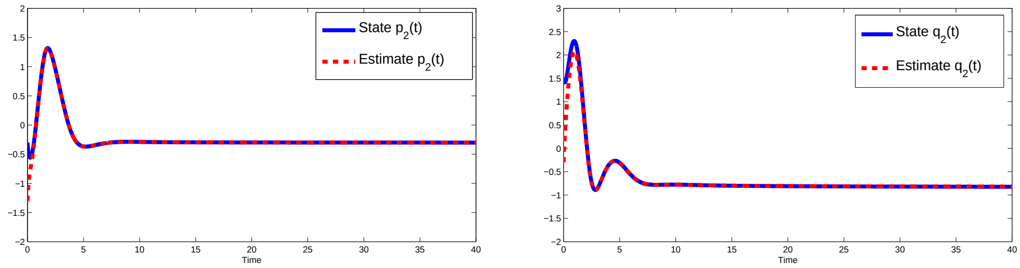

Figure 2: State evolution of $p_2(t)$, $\hat{p}_2(t)$, $q_2(t)$, $\hat{q}_2(t)$ and its estimations with Example 4.1

By means of Assumption (I) *and Assumption* (II)*, we select* $I = diag\{0.2,\ 0.2,\ 0.2\}$, $J = D = diag\{0.5,\ 0.5,\ 0.5\}$, $L = diag\{0.8,\ 0.8,\ 0.8\}$. *Obviously these assumptions are holds. By utilizing the Matlab to solve the LMIs* (10)*, the feasible solutions are given follows,*

$$\Lambda = \begin{bmatrix} 0.1342 & 0.0370 & 0.0654 \\ 0.0370 & 0.1315 & 0.0357 \\ 0.0654 & 0.0357 & 0.2074 \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} 0.4550 & -0.2439 & -0.3056 \\ -0.2324 & 0.6894 & -0.0809 \\ -0.3041 & -0.0846 & 0.5916 \end{bmatrix},$$

$$Y_p = \begin{bmatrix} 0.8406 & -0.0396 & -0.1160 \\ -0.0396 & 0.7910 & -0.0338 \\ -0.1160 & -0.0338 & 0.7382 \end{bmatrix}, \quad Y_q = \begin{bmatrix} 0.0997 & -0.0384 & -0.0513 \\ -0.0384 & 0.1548 & -0.0230 \\ -0.0513 & -0.0230 & 0.1306 \end{bmatrix},$$

$$M = \begin{bmatrix} 8.1375 & -1.7812 & -2.9140 \\ -1.8308 & 6.7204 & -0.6821 \\ -2.8085 & -0.7572 & 4.5945 \end{bmatrix}, \quad N = \begin{bmatrix} 0.4550 & -0.2439 & -0.3056 \\ -0.2324 & 0.6894 & -0.0809 \\ -0.3041 & -0.0846 & 0.5916 \end{bmatrix},$$

$\alpha_1 = 0.6289$, $\alpha_2 = 2.0313$, $\delta = 1.8925$, $\lambda_1 = 3.1802$, $\lambda_2 = 3.3681$ *and* $\mu = 0.3461$.

Therefore it follows from Theorem 3.1, the error system (9) *is globally Mittag-Leffler stable. That is, the system* (8) *becomes a globally Mittag-Leffler state estimator of system* (6)*. Under the initial conditions* $p(t) = \big(3.5, -0.3, 1.75\big)^T$, $\tilde{p}(t) = \big(2, -1.3, 1.5\big)^T$, $q(t) = \big(-0.5, 1.5, 1.75\big)^T$ *and* $\tilde{q}(t) = \big(0.5, -0.3, 4.75\big)^T$, $t \in [-1, 0]$*, the state evolution for each variable of the considered systems* $p_j(t)$, $\hat{p}_j(t)$, $q_k(t)$, $\hat{q}_k(t)$ $(j = k = 1, 2, 3)$ *are illustrated in Fig.[1]-Fig.[3]. Fig.[4] depict the estimator error states of* $z_{pj}(t)$, $z_{qk}(t)$ $(j = k = 1, 2, 3)$*, it notice that the corresponding error states tends to zero, which ensure the validity of our analytical results.*

Example 4.2 *In system* (6)*, we choose the parameter* $\lambda = 0.98$, $\eta = 0.1$, $g(q(t)) = \tanh(q(t))$, $h(p(t)) =$

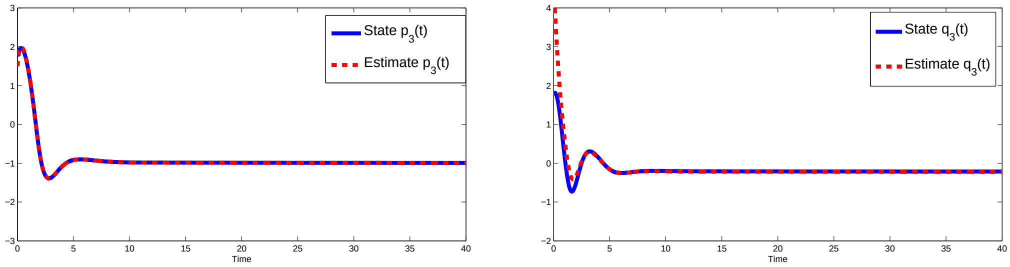

Figure 3: State evolution of $p_3(t),\ \hat{p}_3(t),\ q_3(t),\ \hat{q}_3(t)$ and its estimations with Example 4.1

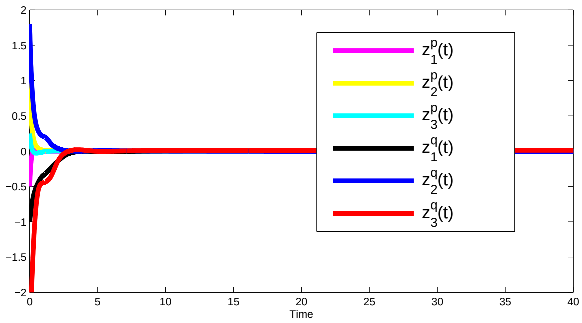

Figure 4: State error evolution of $z_p(t),\ z_q(t)$ and its estimations in Example 4.1

$\tanh(q(t)),\ G = H = [0,\ 0,\ 0]^T,\ A = diag\{2,\ 2,\ 2\},\ U = \{1.5,\ 1.5,\ 1.5\},$

$$
\begin{aligned}
B &= \begin{bmatrix} 2 & 2 & -4.5 \\ -3.75 & 1.51 & -1.1 \\ 1.1 & -3 & 1.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1.5 & 2.5 & -0.4 \\ 1.75 & 0.5 & 1 \\ 2.5 & -2.5 & 0.8 \end{bmatrix}, \\
V &= \begin{bmatrix} 2.1 & 1 & -2 \\ 1.8 & 1.7 & 1 \\ 1.2 & -2.1 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 2.5 & 1.8 \\ -2.2 & 1.6 & 0.2 \\ 1 & 0.4 & 1 \end{bmatrix}.
\end{aligned}
$$

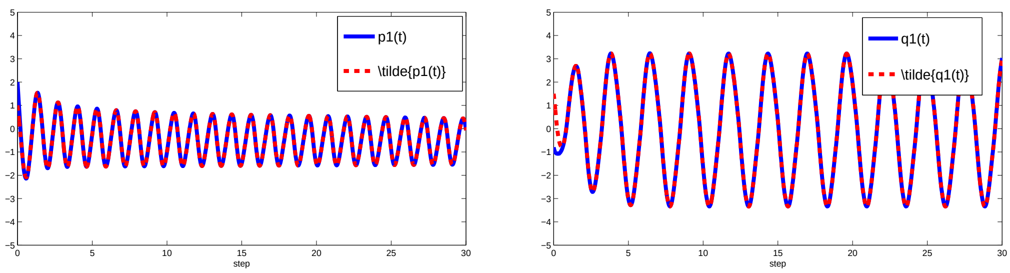

Figure 5: State response of $\big(p_1(t),\ \tilde{p}_1(t)\big)$ and $\big(q_1(t),\ \tilde{q}_1(t)\big)$ in Example 4.2

It is simply to get $L = diag\{0.1,\ 0.1,\ 0.1\},\ D = diag\{0.2,\ 0.2,\ 0.2\}$. *Therefore Assumption* (III) *and Assumption* (IV) *are holds. Now we select* $\pi_j = 0.6,\ \rho_j = 0.8,\ \varepsilon_k = 0.5$ *and* $\varrho_k = 0.5$. *Then the*

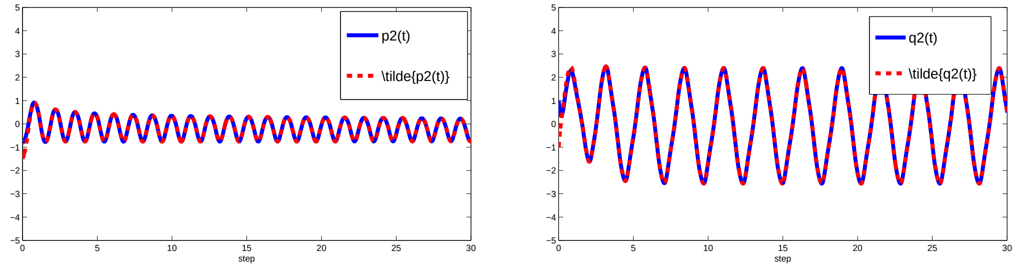

Figure 6: State response of $\big(p_2(t),\ \tilde{p}_2(t)\big)$ and $\big(q_2(t),\ \tilde{q}_2(t)\big)$ in Example 4.2

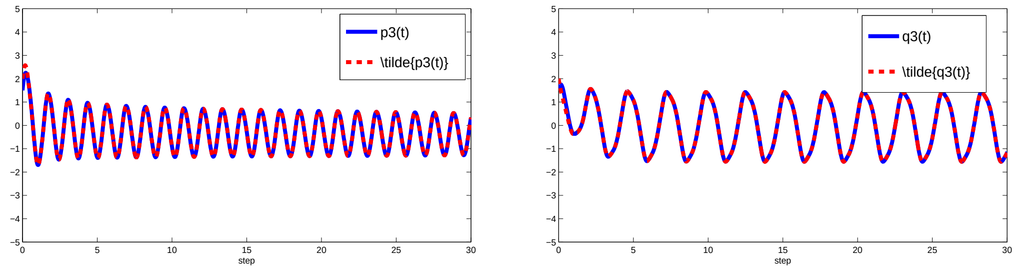

Figure 7: State response of $\big(p_3(t),\ \tilde{p}_3(t)\big)$ and $\big(q_3(t),\ \tilde{q}_3(t)\big)$ in Example 4.2

adaptive feedback controller (19) *is*

$$\begin{cases} \delta_j(t) = -\alpha_j(t)e_{pj}(t) - \mu_j(t)\,\mathrm{sgn}[e_{pj}(t)], \\ \theta_k(t) = -\phi_k(t)e_{qk}(t) - \xi_k(t)\,\mathrm{sgn}[e_{qk}(t)] \end{cases} \tag{33}$$

and the adaptive updated law is

$$\begin{cases} D^{\lambda}\alpha_j(t) = \sum_{k=1}^{3} 0.6e_{pk}(t)e_{pj}(t),\ D^{\lambda}\mu_j(t) = 0.8|e_{pj}(t)|, \\ D^{\lambda}\phi_k(t) = \sum_{j=1}^{3} 0.5e_{qj}(t)\varepsilon_k e_{qk}(t),\ D^{\lambda}\mu_k(t) = 0.5|e_{qk}(t)|, \end{cases}$$

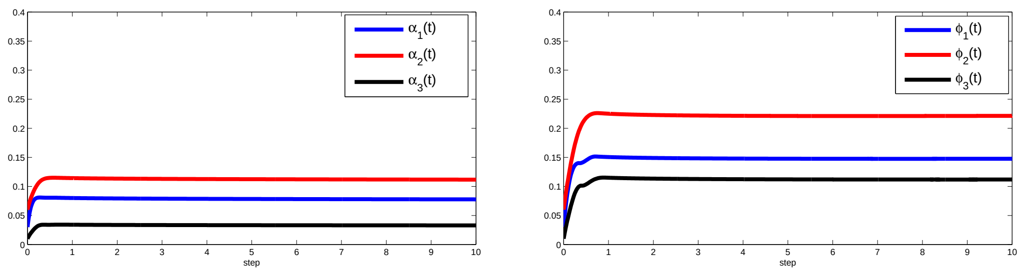

Figure 8: State response of the control gains $\alpha(t)$ and $\phi(t)$ in Example 4.2

for $j = k = 1, 2, 3$. *Take the values of* $r_1 = 0.8,\ r_2 = 0.6,\ \alpha_1 = \alpha_2 = \alpha_3 = 3.8,\ \mu_1 = \mu_2 = \mu_3 = 3,\ \phi_1 = \phi_2 = \phi_3 = 2,\ \xi_1 = 4,\ \xi_2 = 5$ *and* $\xi_3 = 3$. *By means of LMI MATLAB control toolbox, it is easily to obtain the LMI is feasible and the feasible solution is as follows:*

$$\Lambda = 10^8 \times \begin{bmatrix} 8.7446 & 0 & 0 \\ 0 & 8.7446 & 0 \\ 0 & 0 & 8.7446 \end{bmatrix} \quad \Upsilon = 10^9 \times \begin{bmatrix} 1.4757 & 0 & 0 \\ 0 & 1.4757 & 0 \\ 0 & 0 & 1.4757 \end{bmatrix}.$$

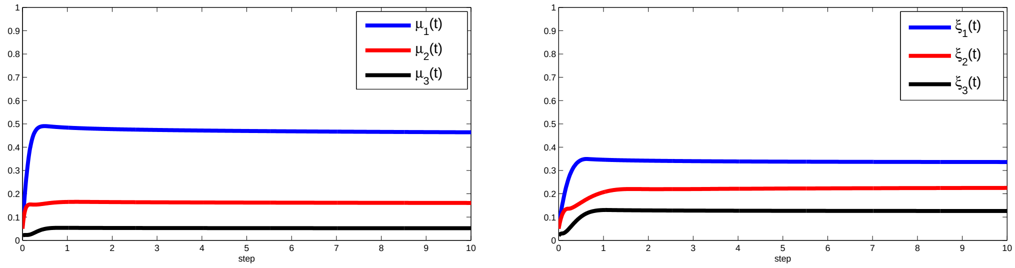

Figure 9: State response of the control gains $\mu(t)$ and $\xi(t)$ in Example 4.2

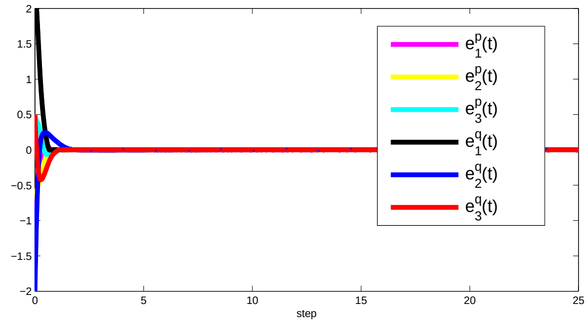

Figure 10: Synchronization error evolution with control inputs in Example 4.2

And after by direct manipulation, it is simply to check that

$$
\begin{aligned}
3 &= \mu_1 > \sum_{k=1}^{3} 2L_k\big(|b_{kj}|+|c_{kj}|\big) = 2.9,\ 2.5 = \mu_2 > \sum_{k=1}^{3} 2L_k\big(|b_{kj}|+|c_{kj}|\big) = 2.4,\\
2 &= \mu_3 > \sum_{k=1}^{3} 2L_k\big(|b_{kj}|+|c_{kj}|\big) = 1.6,\ 4 = \xi_1 > \sum_{j=1}^{n} 2D_j\big(|v_{jk}|+|w_{jk}|\big) = 3.7\\
5 &= \xi_2 > \sum_{j=1}^{n} 2D_j\big(|v_{jk}|+|w_{jk}|\big) = 4.6,\ 3 = \xi_3 > \sum_{j=1}^{n} 2D_j\big(|v_{jk}|+|w_{jk}|\big) = 2.5.
\end{aligned}
$$

Therefore all conditions of Theorem 3.4 are holds. Let the initial conditions of (5) *and* (16) *be* $p(t) = \big(2,-0.8,1.5\big)^T$, $\tilde{p}(t) = \big(1,-1.5,2\big)^T$, $q(t) = \big(-0.8,1,1.5\big)^T$ *and* $\tilde{q}(t) = \big(1.5,-1,2\big)^T$, $t \in [-0.1,0]$. *In Fig.[5]-Fig.[7] displays the evolutions of each variable of the considered systems* $p_j(t)$, $\tilde{p}_j(t)$, $q_k(t)$ *and* $\tilde{q}_k(t)$ $(j = k = 1,2,3)$. *Then, the initial values of the control inputs* (33) *are* $\alpha_1(0) = 0.03$, $\alpha_2(0) = 0.06$, $\alpha_3(0) = 0.01$, $\mu_1(0) = 0.03$, $\mu_2(0) = 0.06$, $\mu_3(0) = 0.01$, $\phi_1(0) = 0.06$, $\phi_2(0) = 0.05$, $\phi_3(0) = 0.02$, $\xi_1(0) = 0.06$, $\xi_2(0) = 0.05$ *and* $\xi_3(0) = 0.02$. *Further it should be mentioned that for the above values, the adaptive coupling strengths* $\alpha_j(t)$, $\mu_j(t)$, $\phi_k(t)$ *and* $\xi_k(t)$ $(j = k = 1,2,3)$ *are shown in Fig.[8]-Fig.[9], it is observed that the adaptive control gains may tends to some positive constants. In Fig.[10] describes time responses synchronization errors* e_{pj} *and* e_{qk} $(j = k = 1,2,3)$. *It is notice that the synchronization errors converges to zero, which confirms the effectiveness of our results. Hence, these simulations results indicates the slave system* (16) *are globally Mittag-Leffler synchronized with the master system* (5) *under the controller* (33).

Remark 4.3 *The authors in [48] designed the state feedback controller* $\delta(t) = -Ke_p(t)$ *and* $\theta(t) =$

$-\tilde{K}e_q(t)$ to the slave system, which control gain is denoted by $K = diag\{14, 14\}$ and $\tilde{K} = diag\{16, 16\}$, respectively. The authors in [5] utilizing the adaptive feedback controller $\delta(t) = -\zeta(t)e_p(t) - \eta(t)\,\mathrm{sgn}(e_p(t))$ and $\theta(t) = -\alpha(t)e_p(t) - \beta(t)\,\mathrm{sgn}(e_q(t))$ to the slave system and obtained final control gains are $\zeta(t) \leq 2.1$, $\eta(t) \leq 2.8$, $\alpha(t) \leq 1.2$ and $\beta(t) \leq 1.25$. But, our control gains $\alpha(t) \leq 0.112$, $\phi(t) \leq 0.222$, $\mu(t) \leq 0.471$ and $\xi(t) \leq 0.341$ in example 4.2 are much smaller than control gains of above mentioned references. Hence, example 4.2 shown that the designed adaptive controller in FBNNs is more effective comparing to Ref [5, 48]. However, the adaptive Mittag-Leffler synchronization of FBNNs have not yet been seen, hence the result is new. Moreover, in example 4.2 we deal with the adaptive synchronization of FBNNs via Theorem 3.8. If memristive connection weights are invariable in Ref [33], it is worth pointing that the numerical examples of integer order BAM neural networks in [33] can be the special cases of example 4.2.

5 Conclusions

In this paper, we devoted to demonstrate the issues of the extended design of Mittag-Leffler state estimator and synchronization for FBNNs with time delay. By a key role of the Razumikhin-type method, some new sufficient criteria ensuring Mittag-Leffler state estimator of the proposed model are investigated in terms of LMI approach. Moreover, a novel adaptive feedback controller is designed. By using this controller, algebraic sufficient conditions are obtained to guarantee the Mittag-Leffler synchronization. Again, the corollaries had been given to demonstrate the obtained theoretical outcomes within the paper are also authentic for FBNNs without delay term. Lastly, two simulation examples affirm the rationality of the theoretical results. It would be interesting to extend the results proposed in this paper to the state estimator and synchronization analysis for fractional order memristor based Cohen-Grossberg BAM neural networks with discrete time delays, which will be considered in our future research.

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